Hardy algebras associated with W^*-correspondences (point evaluation and Schur class functions)

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1   Introduction

This is primarily an exposition of our work in [32] and [34] which builds on the theory of tensor algebras over C^*-correspondences that we developed in [28]. Operator tensor algebras (and their w^*-analogues, which we call *Hardy*

algebras) form a rich class of non-self-adjoint operator algebras that contains a large variety of operator algebras that have received substantial attention in the literature in recent years.

Among these algebras are the classical disc algebra $A(\mathbb{D})$, and its weak closure, $H^\infty(\mathbb{T})$; Popescu's non commutative disc algebras [42], and their weak closures, the free semigroup algebras studied by Popescu [42] and Davidson and Pitts [17]; quiver algebras studied by us in [29] and by Kribs and Power in [22]; certain nest algebras; analytic crossed products, studied by Peters [38] and by McAsey and Muhly in [25]; and others. (We will describe the construction of tensor and Hardy algebras and give many examples in Section 2.) The theory gives a common approach to the analysis of all these algebras and has its roots deeply embedded in the model theory of contraction operators on the one hand and in classical ring theory on the other.

In fact, the theory of contraction operators may be viewed as the theory of contractive representations of the disc algebra. The representation theory of the tensor algebras is a natural generalization of this theory that preserves many of its features. The disc algebra may be viewed as an analytic generalization of the polynomial algebra in one variable. The interplay between function theory and the representation theory of the polynomial algebra has been one of the guiding beacons in model theory for decades [15]. The tensor algebras we analyze are operator algebraic versions of algebras that generalize polynomial algebras and have been of key importance in ring theory since 1947 [20] and, since 1972, have been a major focus of attention for analyzing finite dimensional algebras (see [18] and [19]). (See [27] for an extended discussion of the connection between operator tensor algebras and the theory of finite dimensional algebras.)

Recall that the disc algebra $A(\mathbb{D})$ may be realized as the algebra of all analytic Toeplitz operators on $l^2(\mathbb{N})$ (or on $H^2(\mathbb{T})$). Popescu generalizes $A(\mathbb{D})$ by considering algebras of operators on the full Fock space over a Hilbert space H of some dimension, n say. Let $\mathcal{F}(H) = \mathbb{C} \oplus H \oplus H^{\otimes 2} \oplus \cdots$ denote this Fock space. Then his noncommutative disc algebra of index n is the norm closed algebra generated by the (left) creation operators. That is, his algebras are generated by the identity operator and operators of the form $\lambda(\xi)\eta := \xi \otimes \eta$, where $\xi \in H$ and $\eta \in \mathcal{F}(H)$. The Fock space may also be written as $l^2(\mathbb{F}_+^n)$ where $\mathbb{F}_+^n$ is the free semigroup on n generators. In this realization, H may be identified with all the functions supported on the words of length one and for such a function ξ, $\lambda(\xi)$ is just convolution by ξ on $l^2(\mathbb{F}_+^n)$. Observe that when n, the dimension of H, is one, then one

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recovers the disc algebra $A(\mathbb{D})$ represented by analytic Toeplitz operators on $l^2(\mathbb{N})$.

To construct more general tensor algebras one replaces the Hilbert space H by a correspondence over some C^*-algebra (or von Neumann algebra) M. Roughly, a correspondence is a bimodule over M that is also equipped with an M-valued inner product. (For a precise definition see Section 2). When $M = \mathbb{C}$ a correspondence over M is just a Hilbert space.

The tensor algebra associated with E, $\mathcal{T}_+(E)$, is generated by creation operators on the Fock space $\mathcal{F}(E) = M \oplus E \oplus E^{\otimes 2} \cdots$ together with a copy of M (formed by diagonal operators of multiplication, $\varphi_\infty(a)$, $a \in M$). It follows from the results of [28] that (completely contractive) representations of $\mathcal{T}_+(E)$ are given by pairs (T, σ) where $T : E \to B(H)$ is a completely contractive map and $\sigma : M \to B(H)$ is a C^*-representation of M that satisfy $T(a \cdot \xi \cdot b) = \sigma(a)T(\xi)\sigma(b)$ for $a, b \in M$ and $\xi \in E$. (Note that we shall sometimes use φ for the left multiplication on E; that is, $a \cdot \xi$ may be written $\varphi(a)\xi$). Such pairs, (T, σ), are called *covariant representations* of the correspondence E. Given (T, σ), one may form the Hilbert space $E \otimes_\sigma H$ (see the discussion following Definition 2.1). For $a \in M$, $\varphi(a) \otimes I$ then defines a bounded operator on this space. The "complete contractivity" of T is equivalent to the assertion that the linear map $\tilde{T}$ defined initially on the balanced algebraic tensor product $E \otimes H$ by the formula $\tilde{T}(\xi \otimes h) := T(\xi)h$ extends to an operator of norm at most 1 on the completion $E \otimes_\sigma H$. The bimodule property of T, then, is equivalent to the equation

$$\tilde{T}(\varphi(a) \otimes I) = \sigma(a)\tilde{T}, \tag{1}$$

for all $a \in M$, which means that $\tilde{T}$ intertwines σ and the natural representation of M on $E \otimes_\sigma H$ - the composition of φ with Rieffel's induced representation of $\mathcal{L}(E)$ determined by σ.

Thus we see that, once σ is fixed, the representations of $\mathcal{T}_+(E)$ are parameterized by the elements in the closed unit ball of the intertwining space $\{\eta \in B(E \otimes_\sigma H, H) \mid \eta(\varphi(\cdot) \otimes I) = \sigma\eta$ and $\|\eta\| \leq 1\}$. Reflecting on this leads one ineluctably to the functional analyst's imperative: *To understand an algebra, view it as an algebra of functions on its space of representations.* In our setting, then, we want to think about $\mathcal{T}_+(E)$ as a space of functions on this ball. For reasons that will be revealed in a minute, we prefer to focus on the adjoints of the elements in this space. Thus we let $E^\sigma = \{\eta \in B(H, E \otimes_\sigma H) \mid \eta\sigma = (\varphi(\cdot) \otimes I)\eta\}$ and we write $\mathbb{D}((E^\sigma)^*)$ for the

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set $\{\eta \in B(E \otimes_\sigma H, H) \mid \eta^* \in E^\sigma, \text{ and } \|\eta\| < 1\}$. That is, $\mathbb{D}((E^\sigma)^*)$ is the norm-interior of the representation space consisting of those (T, σ) that are "anchored by σ". One of our interests, then, is to understand the kind of functions that elements X of $\mathcal{T}_+(E)$ determine on $\mathbb{D}((E^\sigma)^*)$ via the formula

$$X(\eta^*) = \sigma \times \eta^*(X),$$

where $\sigma \times \eta^*$ is the representation of $\mathcal{T}_+(E)$ that is determined by the pair (σ, T) with $\tilde{T} = \eta^*$.

In the special case when $A = E = \mathbb{C}$ and σ is the one-dimensional representation of A on $\mathbb{C}$, E^σ is also one-dimensional, so $\mathbb{D}((E^\sigma)^*)$ is just the open unit disc in the complex plane and, for $X \in \mathcal{T}_+(E)$, $X(\eta^*)$ is the ordinary value of X at the complex number $\bar{\eta}$. On the other hand, if $A = E = \mathbb{C}$, and if σ is scalar multiplication on a Hilbert space H (the only possible representation of $\mathbb{C}$ on H), then $\mathbb{D}((E^\sigma)^*)$ is the space of *strict* contraction operators on H and for $\eta^* \in \overline{\mathbb{D}((E^\sigma)^*)}^{\|\cdot\|}$ and $X \in \mathcal{T}_+(E) = A(\mathbb{D})$, $X(\eta^*)$ is simply the value of X at η^* defined through the Sz.-Nagy–Foiaş functional calculus [36]. For another example, if $A = \mathbb{C}$, but $E = \mathbb{C}^n$, and if σ is scalar multiplication on a Hilbert space H, then $\mathbb{D}((E^\sigma)^*)$ is the space of row contractions on H, $(T_1, T_2, \cdots, T_n)$, of norm less than 1; i.e. $\sum T_i^* T_i \le rI_H$ for some $r < 1$. In this case, $X(\eta^*)$ is given by Popescu's functional calculus [43].

In addition to parametrizing certain representations of $\mathcal{T}_+(E)$, E^σ has another fundamental property: It is itself a C^*-correspondence - over *the von Neumann algebra* $\sigma(A)'$. Indeed, it is not difficult to see that E^σ becomes a bimodule over $\sigma(A)'$ via the formulae: $a \cdot \eta = (I_E \otimes a)\eta$ and $\eta \cdot a = \eta a$, $\eta \in E^\sigma$, $a \in \sigma(A)'$. Further, if η and ζ are in E^σ, then the product $\eta^* \zeta$ lies in the commutant $\sigma(A)'$ and defines a $\sigma(A)'$-valued inner product $\langle \eta, \zeta \rangle$ making E^σ a C^*-correspondence. In fact, since E^σ is a weakly closed space of operators, it has certain topological properties making it what we call a W^*-correspondence [32]. It is because E^σ is a W^*-correspondence over $\sigma(A)'$ that we focus on it, when studying representations of $\mathcal{T}_+(E)$, rather than on its space of adjoints. While E^σ plays a fundamental role in our study of quantum Markov processes [30], its importance here - besides providing a space on which to "evaluate" elements of $\mathcal{T}_+(E)$ - lies in the fact that a certain natural representation of E^σ generates the commutant of the representation of $\mathcal{T}_+(E)$ obtained by "inducing σ up to" $\mathcal{L}(\mathcal{F}(E))$. (See Theorem 2.29).

It is primarily because of this commutant theorem that we cast our work in this paper entirely in terms of W^*-correspondences. That is, we work with

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von Neumann algebras M and W^*-correspondences E over them. We still form the Fock space $\mathcal{F}(E)$ and the tensor algebra $\mathcal{T}_+(E)$ over E, but because $\mathcal{F}(E)$ is a W^*-correspondence over M, the space $\mathcal{L}(\mathcal{F}(E))$ is a von Neumann algebra. We call the w^*-closure of $\mathcal{T}_+(E)$ in $\mathcal{L}(\mathcal{F}(E))$ the *Hardy algebra* of E and denote it by $H^\infty(E)$. This is our principal object of study. In the case when $M = E = \mathbb{C}$, $H^\infty(E)$ it the classical $H^\infty(\mathbb{T})$ (viewed as analytic Toeplitz operators).

As we will see in Lemma 2.17, given a faithful normal representation σ of M on a Hilbert space H, we may also evaluate elements in $H^\infty(E)$ at points in $\mathbb{D}((E^\sigma)^*)$ (since the representation associated with a point in the open unit ball extends from $\mathcal{T}_+(E)$ to $H^\infty(E)$). That is, elements in $H^\infty(E)$ may be viewed as functions on $\mathbb{D}((E^\sigma)^*)$, also. Further, when $H^\infty(E)$ is so represented, one may study the "value distribution theory" of these functions. In this context, we establish two capstone results from function theory: The first, [32, Theorem 5.3] is presented as Theorem 3.2 below, generalizes the Nevanlinna-Pick interpolation theorem. It asserts that given two k-tuples of operators in $B(H)$ (where H is the representation space of σ), $B_1, B_2, \cdots, B_k$, and $C_1, C_2, \cdots, C_k$, and given points $\eta_1, \eta_2, \cdots, \eta_k$ in $\mathbb{D}((E^\sigma)^*)$, one may find an element X in $H^\infty(E)$ of norm at most one such that

$$B_i X(\eta_i^*) = C_i,$$

for all i if and only if a certain matrix of *maps*, which resembles the classical Pick matrix, represents a completely positive operator. This result captures numerous theorems in the literature that go under the name of generalized Nevanlinna-Pick theorems. Our proof of the theorem (in [32]) uses a commutant lifting theorem that we proved in [28]. In the context of model theory, it was Sarason who introduced the use of commutant lifting to prove the interpolation theorem ([46]). More recently, a number of authors have been studying interpolation problems in the context of reproducing kernel Hilbert spaces. (See [12] and [1]).

Our second capstone result is a generalization of Schwartz's lemma (see Theorem 3.4). It follows from our Nevanlinna-Pick theorem that an element X in $H^\infty(E)$ of norm at most one defines a "Pick-type" matrix of maps that represents a completely positive map. In fact, the matrix is defined using the values of X on $\mathbb{D}((E^\sigma)^*)$. Given an arbitrary operator-valued function Z on $\mathbb{D}((E^\sigma)^*)$, one may define a matrix of maps in a similar way. We say that Z is a *Schur class operator function* if this matrix defines a completely

positive map. (See Definition 4.2 for a precise statement). Theorem 3.2 then shows that the function $\eta^* \mapsto X(\eta^*)$ is a Schur class operator function for X in the closed unit ball of $H^\infty(E)$. In fact, we show in Theorem 4.3 that every Schur class operator function arises in this way and that every such function (with values in, say, $B(\mathcal{E})$) may be represented in the form $Z(\eta^*) = A + B(I - L_\eta^* D)^{-1} L_\eta^* C$ where A, B, C and D are the entries of a 2×2 block matrix representing a coisometric operator V from $\mathcal{E} \oplus H$ to $\mathcal{E} \oplus (E^\sigma \otimes H)$ (for some auxiliary Hilbert space H) with a certain intertwining property and L_η is the operator from H to $E^\sigma \otimes H$ that maps h to $\eta \otimes h$. Borrowing terminology from the classical function theory on the unit disc $\mathbb{D}$, we call such a representation a *realization* of Z and we call the coisometry V a *colligation*. (In general, V is a coisometry but, under a mild assumption, it may be chosen to be unitary.)

These results, together with our work on canonical models in [33], represent a generalization of some of the essential ingredients of a program that has been developed successfully in model theory - the interaction between operator theory and function theory on the unit disc $\mathbb{D}$ - and has been generalized in various ways to the polydisc and the ball in $\mathbb{C}^n$. This program sets up (essentially) bijective correspondences connecting the theory of unitary colligations (and their unitary systems), the Sz-Nagy-Foias functional model theory for contraction operators and the discrete-time Lax-Phillips scattering theory. Each theory produces a contractive operator-valued function (called the transfer function of the system, the characteristic operator function of the completely non unitary contraction or the scattering function for the scattering system) from which one can recover the original object (the system or the contraction) up to unitary equivalence. For more details, see the works of Ball ([10]), Ball and Vinnikov ([13]), Ball, Trent and Vinnikov ([12]) and the references there.

We shall not discuss the program in detail here but we note that Theorem 4.3 below is the generalization, to our context, of Theorem 2.1 of [10] or Theorem 2.1 of [12]. Here the elements of $H^\infty(E)$ play the role of multipliers and the disc $\mathbb{D}$ in $\mathbb{C}$ is replaced by the open unit ball of $(E^\sigma)^*$.

We also note that the canonical models for contraction operators are replaced, in our setting, by canonical models for representations of $H^\infty(E)$. This theory was developed in [33] for completely noncoisometric representations (generalizing results of Popescu in [41]) and it is shown there that the characteristic operator function for such a representation has a realization associated with a unitary colligation.

In the next section we set the stage by defining our basic constructions, presenting examples and emphasizing the roles of duality and point evaluation in the theory.

Section 3 deals with the Nevanlinna-Pick theorem and Section 4 with Schur class operator functions.

2 Preliminaries: W^*-correspondences and Hardy algebras

We shall follow Lance [24] for the general theory of Hilbert C^*-modules that we use. Let A be a C^*-algebra and let E be a right module over A endowed with a bi-additive map $\langle \cdot, \cdot \rangle : E \times E \to A$ (referred to as an A-valued inner product) such that, for $\xi, \eta \in E$ and $a \in A$, $\langle \xi, \eta a \rangle = \langle \xi, \eta \rangle a$, $\langle \xi, \eta \rangle^* = \langle \eta, \xi \rangle$, and $\langle \xi, \xi \rangle \geq 0$, with $\langle \xi, \xi \rangle = 0$ only when $\xi = 0$. If E is complete in the norm $\|\xi\| := \|\langle \xi, \xi \rangle\|^{1/2}$, the E is called a (right) *Hilbert C^*-module* over A. We write $\mathcal{L}(E)$ for the space of continuous, adjointable, A-module maps on E; that is every element of $\mathcal{L}(E)$ is continuous and if $X \in \mathcal{L}(E)$, then there is an element $X^* \in \mathcal{L}(E)$ that satisfies $\langle X^*\xi, \eta \rangle = \langle \xi, X\eta \rangle$. The element X^* is unique and $\mathcal{L}(E)$ is a C^*-algebra with respect to the involution $X \to X^*$ and the operator norm. If M is a von Neumann algebra and if E is a Hilbert C^*-module over M, then E is said to be *self dual* in case every continuous M-module map from E to M is given by an inner product with an element of E. If E is a self dual Hilbert module over M, then $\mathcal{L}(E)$ is a W^*-algebra and coincides with *all* the bounded linear maps on E [37].

A C^*-*correspondence* over a C^*-algebra A is a Hilbert C^*-module E over A endowed with a structure of a left module over A via a *-homomorphism $\varphi : A \to \mathcal{L}(E)$. When dealing with a specific C^*-correspondence E over a C^*-algebra A, it will be convenient to suppress the φ in formulas involving the left action and simply write $a\xi$ or $a \cdot \xi$ for $\varphi(a)\xi$. This should cause no confusion in context.

Having defined a left action on E, we are allowed to form balanced tensor products. Given two correspondences E and F over the C^*-algebra A one may define an A-valued inner product on the balanced tensor product $E \otimes_A F$ by the formula

$$\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle_{E \otimes_B F} := \langle \eta_1, \varphi(\langle \xi_1, \xi_2 \rangle_E)\eta_2 \rangle_F.$$

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The Hausdorff completion of this bimodule is again denoted by $E \otimes_A F$ and is called the *tensor product* of E and F.

Definition 2.1 *Let M be a von Neumann algebra and let E be a Hilbert C^*-module over M. Then E is called a* Hilbert W^*-module *over M in case E is self-dual. The module E is called a W^*-correspondence over M in case E is a self-dual C^*-correspondence over M such that the $*$-homomorphism $\varphi : M \to \mathcal{L}(E)$ giving the left module structure on E is normal.*

It is evident that the tensor product of two W^*-correspondences is again a W^*-correspondence. Note also that, given a W^*-correspondence E over M and a Hilbert space H equipped with a normal representation σ of M, we may form the Hilbert space $E \otimes_\sigma H$ (by defining $\langle \xi_1 \otimes h_1, \xi_2 \otimes h_2 \rangle = \langle h_1, \sigma(\langle \xi_1, \xi_2 \rangle) h_2 \rangle$). Then, given an operator $X \in \mathcal{L}(E)$ and an operator $S \in \sigma(N)'$, the map $\xi \otimes h \mapsto X\xi \otimes Sh$ defines a bounded operator on $E \otimes_\sigma H$ denoted by $X \otimes S$. When $S = I$ and $X = \varphi(a)$, $a \in M$, we get a representation of M on this space.

Observe that if E is a W^*-correspondence over a von Neumann algebra M, then each of the tensor powers of E, viewed as a C^*-correspondence over M in the usual way, is in fact a W^*-correspondence over M and so, too, is the full Fock space $\mathcal{F}(E)$, which is defined to be the direct sum $M \oplus E \oplus E^{\otimes 2} \oplus \cdots$, with its obvious structure as a right Hilbert module over M and left action given by the map φ_∞, defined by the formula $\varphi_\infty(a) := diag\{a, \varphi(a), \varphi^{(2)}(a), \varphi^{(3)}(a), \cdots\}$, where for all n, $\varphi^{(n)}(a)(\xi_1 \otimes \xi_2 \otimes \cdots \xi_n) = (\varphi(a)\xi_1) \otimes \xi_2 \otimes \cdots \xi_n$, $\xi_1 \otimes \xi_2 \otimes \cdots \xi_n \in E^{\otimes n}$. The *tensor algebra* over E, denoted $\mathcal{T}_+(E)$, is defined to be the norm-closed subalgebra of $\mathcal{L}(\mathcal{F}(E))$ generated by $\varphi_\infty(M)$ and the *creation operators* T_ξ, $\xi \in E$, defined by the formula $T_\xi \eta = \xi \otimes \eta$, $\eta \in \mathcal{F}(E)$. We refer the reader to [28] for the basic facts about $\mathcal{T}_+(E)$.

Definition 2.2 *Given a W^*-correspondence E over the von Neumann algebra M, the ultraweak closure of the tensor algebra of E, $\mathcal{T}_+(E)$, in the w^*-algebra $\mathcal{L}(\mathcal{F}(E))$, will be called the* Hardy Algebra *of E, and will be denoted $H^\infty(E)$.*

Example 2.3 *If $M = E = \mathbb{C}$ then $\mathcal{F}(E)$ may be identified with $H^2(\mathbb{T})$. The tensor algebra in this setting is isomorphic to the disc algebra $A(\mathbb{D})$ and the Hardy algebra is the classical Hardy algebra $H^\infty(\mathbb{T})$.*

Example 2.4 If $M = \mathbb{C}$ and $E = \mathbb{C}^n$, then $\mathcal{F}(E)$ may be identified with the space $l_2(\mathbb{F}_n^+)$ where $\mathbb{F}_n^+$ is the free semigroup on n generators. The tensor algebra then is what Popescu refers to as the "non commutative disc algebra" $\mathcal{A}_n$ and the Hardy algebra is its w^*-closure. It was studied by Popescu ([44]) and by Davidson and Pitts who denoted it by $\mathcal{L}_n$ ([17]).

Example 2.5 Let M be a von Neumann algebra and let α be a unital, injective, normal *-endomorphism on M. The correspondence E associated with α is equal to M as a vector space. The right action is by multiplication, the M-valued inner product is $\langle a, b \rangle = a^*b$ and the left action is given by α; i.e. $\varphi(a)b = \alpha(a)b$. We write $_\alpha M$ for E. It is easy to check that $E^{\otimes n}$ is isomorphic to $_{\alpha^n} M$. The Hardy algebra in this case is called the non-selfadjoint crossed product of M by α and will be written $M \rtimes_\alpha \mathbb{Z}_+$. This algebra is also called an analytic crossed product, at least when α is an automorphism. It is related to the algebras studied in [25] and [38]. If we write w for T_1 (where 1 is the identity of M viewed as an element of E), then the algebra is generated by w and $\varphi_\infty(M)$ and every element X in the algebra has a formal "Fourier" expression

$$X = \sum_{n=0} w^n b_n$$

where $b_n \in \varphi_\infty(M)$. This Fourier expression is actually Ceasaro-summable to X in the ultraweak topology on $H^\infty(E)$ [32], but we don't need these details in the present discussion.

Example 2.6 Here we set M to be the algebra $l_\infty(\mathbb{Z})$ and let α be the automorphism defined by $(\alpha(g))_i = g_{i-1}$. Write E for the correspondence $_\alpha M$ as in Example 2.5. Another, isomorphic, way to describe E is to let M be the algebra $\mathcal{D}$ of all diagonal operators on $l^2(\mathbb{Z})$, let U be the shift defined by $Ue_k = e_{k-1}$ (where $\{e_k\}$ is the standard basis), and set $E = U\mathcal{D} \subseteq B(l^2(\mathbb{Z}))$. The left and right actions on E are defined simply by operator multiplications and the inner product is $\langle UD_1, UD_2 \rangle = D_1^*D_2$. It is easy to check that these correspondences are indeed isomorphic and the Hardy algebra $H^\infty(E)$ is (completely isometrically isomorphic to) the algebra $\mathcal{U}$ of all operators in $B(l_2(\mathbb{Z}))$ whose matrix (with respect to the standard basis) is upper triangular.

Example 2.7 Suppose that Θ is a normal, contractive, completely positive map on a von Neumann algebra M. Then we may associate with it the

correspondence $M \otimes_\Theta M$ obtained by defining the M-valued inner product on the algebraic tensor product $M \otimes M$ via the formula $\langle a \otimes b, c \otimes d \rangle = b^* \theta(a^* c) d$ and completing. (The bimodule structure is by left and right multiplications). This correspondence was used by Popa ([40]), Mingo ([26]), Anantharam-Delarouche ([4]) and others to study the map Θ. If Θ is an endomorphism this correspondence is the one described in example 2.5.

Example 2.8 Let M be D_n, the diagonal $n \times n$ matrices and E be the set of all $n \times n$ matrices $A = (a_{ij})$ with $a_{ij} = 0$ unless $j = i + 1$ with the inner product $\langle A, B \rangle = A^* B$ and the left and right actions given by matrix multiplication. Then the Hardy algebra is isomorphic to T_n, the $n \times n$ upper triangular matrices. In fact, a similar argument works to show that, for every finite nest of projections $\mathcal{N}$ on a Hilbert space H, the nest algebra alg$\mathcal{N}$ (i.e. the set of all operators on H leaving the ranges of the projections in $\mathcal{N}$ invariant) may be written as $H^\infty(E)$ for some W^*-correspondence E.

Example 2.9 (Quiver algebras) Let $\mathcal{Q}$ be a directed graph on the set V of vertices. For simplicity we assume that both V and $\mathcal{Q}$ are finite sets and view each $\alpha \in \mathcal{Q}$ as an arrow from $s(\alpha)$ (in V) to $r(\alpha)$ (in V). Let M be $C(V)$ (a von Neumann algebra) and E (or $E(\mathcal{Q})$) be $C(\mathcal{Q})$. Define the M-bimodule structure on E as follows: for $f \in E$, $\psi \in M$ and $\alpha \in \mathcal{Q}$,

$$(f\psi)(\alpha) = f(\alpha)\psi(s(\alpha)),$$

and

$$(\psi f)(\alpha) = \psi(r(\alpha))f(\alpha).$$

The M-valued inner product is given by the formula

$$\langle f, g \rangle(v) = \sum_{s(\alpha)=v} \overline{f(\alpha)} g(\alpha),$$

for $f, g \in E$ and $v \in V$. The algebra $H^\infty(E)$ in this case will be written $H^\infty(\mathcal{Q})$ and is the σ-weak closure of $\mathcal{T}_+(E(\mathcal{Q}))$. Viewing both algebras as acting on the Fock space, one sees that they are generated by a set $\{S_\alpha : \alpha \in \mathcal{Q}\}$ of partial isometries (here $S_\alpha = T_{\delta_\alpha}$ where δ_α is the function in $C(\mathcal{Q})$ which is 1 at α and 0 otherwise) and a set $\{P_v : v \in V\}$ of projections (i.e. the generators of $\varphi_\infty(M)$) satisfying the following conditions.

(i) $P_v P_u = 0$ if $u \neq v$,

(ii) $S_\alpha^* S_\beta = 0$ *if* $\alpha \neq \beta$

(iii) $S_\alpha^* S_\alpha = P_{s(\alpha)}$ *and*

(iv) $\sum_{r(\alpha)=v} S_\alpha S_\alpha^* \leq P_v$ *for all* $v \in V$.

These algebras were studied in [27] and [29], and also in [22] where they were called free semigroupoid algebras.

2.1 Representations

In most respects, the representation theory of $H^\infty(E)$ follows the lines of the representation theory of $\mathcal{T}_+(E)$. However, there are some differences that will be important to discuss here. To help illuminate these, we need to review some of the basic ideas from [28, 29, 32].

A representation ρ of $H^\infty(E)$ (or of $\mathcal{T}_+(E)$) on a Hilbert space H is completely determined by what it does to the generators. Thus, from a representation ρ we obtain two maps: a map $T : E \to B(H)$, defined by $T(\xi) = \rho(T_\xi)$, and a map $\sigma : M \to B(H)$, defined by $\sigma(a) = \rho(\varphi_\infty(a))$. Analyzing the properties of T and σ one is lead to the following definition.

Definition 2.10 *Let E be a W^*-correspondence over a von Neumann algebra M. Then a* completely contractive covariant representation *of E on a Hilbert space H is a pair (T, σ), where*

1. *σ is a normal $*$-representation of M in $B(H)$.*

2. *T is a linear, completely contractive map from E to $B(H)$ that is continuous in the σ-topology of [9] on E and the ultraweak topology on $B(H)$.*

3. *T is a bimodule map in the sense that $T(S\xi R) = \sigma(S)T(\xi)\sigma(R)$, $\xi \in E$, and $S, R \in M$.*

It should be noted that there is a natural way to view E as an operator space (by viewing it as a subspace of its linking algebra) and this defines the operator space structure of E to which the Definition 2.10 refers when it is asserted that T is completely contractive.

As we noted in the introduction and developed in [28, Lemmas 3.4–3.6] and in [32], if a completely contractive covariant representation, (T, σ), of E

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in $B(H)$ is given, then it determines a contraction $\tilde{T} : E \otimes_\sigma H \to H$ defined by the formula $\tilde{T}(\eta \otimes h) := T(\eta)h$, $\eta \otimes h \in E \otimes_\sigma H$. The operator $\tilde{T}$ satisfies

$$\tilde{T}(\varphi(\cdot) \otimes I) = \sigma(\cdot)\tilde{T}. \tag{2}$$

In fact we have the following lemma from [32, Lemma 2.16].

Lemma 2.11 *The map $(T, \sigma) \to \tilde{T}$ is a bijection between all completely contractive covariant representations (T, σ) of E on the Hilbert space H and contractive operators $\tilde{T} : E \otimes_\sigma H \to H$ that satisfy equation (2). Given σ and a contraction $\tilde{T}$ satisfying the covariance condition (2), we get a the completely contractive covariant representation (T, σ) of E on H by setting $T(\xi)h := \tilde{T}(\xi \otimes h)$.*

The following theorem shows that every completely contractive representation of the tensor algebra $\mathcal{T}_+(E)$ is given by a pair (T, σ) as above or, equivalently, by a contraction $\tilde{T}$ satisfying (2).

Theorem 2.12 *([28, Theorem 3.10]) Let E be a W^*-correspondence over a von Neumann algebra M. To every completely contractive covariant representation, (T, σ), of E there is a unique completely contractive representation ρ of the tensor algebra $\mathcal{T}_+(E)$ that satisfies*

$$\rho(T_\xi) = T(\xi) \quad \xi \in E$$

and

$$\rho(\varphi_\infty(a)) = \sigma(a) \quad a \in M.$$

The map $(T, \sigma) \mapsto \rho$ is a bijection between the set of all completely contractive covariant representations of E and all completely contractive (algebra) representations of $\mathcal{T}_+(E)$ whose restrictions to $\varphi_\infty(M)$ are continuous with respect to the ultraweak topology on $\mathcal{L}(\mathcal{F}(E))$.

Definition 2.13 *If (T, σ) is a completely contractive covariant representation of a W^*-correspondence E over a von Neumann algebra M, we call the representation ρ of $\mathcal{T}_+(E)$ described in Theorem 2.12 the integrated form of (T, σ) and write $\rho = \sigma \times T$.*

Example 2.14 *In the context of Example 2.4, $M = \mathbb{C}$ and $E = \mathbb{C}^n$. Then, a completely contractive covariant representation of E is simply given by a completely contractive map $T : E \to B(H)$. Writing $T_k = T(e_k)$, where e_k is the standard basis in $\mathbb{C}^n$, and identifying $\mathbb{C}^n \otimes H$ with the direct sum of n copies of H, we may write $\tilde{T}$ as a row $(T_1, T_2, \ldots, T_n)$. The condition that $\|\tilde{T}\| \leq 1$ is the condition (studied by Popescu [42] and Davidson and Pitts [17]) that $\sum T_i T_i^* \leq 1$. Hence representations of the noncommutative disc algebras are given by row contractions.*

Example 2.15 *Consider the setting of Example 2.9 and let $V, \mathcal{Q}, M$ and E be as defined there. A (completely contractive covariant) representation of E is given by a representation σ of $M = C(V)$ on a Hilbert space H and by a contractive map $\tilde{T} : E \otimes_\sigma H \to H$ satisfying (2) above. Write δ_v for the function in $C(V)$ which is 1 on v and 0 elsewhere. The representation σ is given by the projections $Q_v = \sigma(\delta_v)$ whose sum is I. For every $\alpha \in \mathcal{Q}$ write δ_α for the function (on E) which is 1 at α and 0 elsewhere. Given $\tilde{T}$ as above, we may define maps $T(\alpha) \in B(H)$ by $T(\alpha)h = \tilde{T}(\delta_\alpha \otimes h)$ and it is easy to check that $\tilde{T}\tilde{T}^* = \sum_\alpha T(\alpha)T(\alpha)^*$ and $T(\alpha) = Q_{r(\alpha)}T(\alpha)Q_{s(\alpha)}$. Thus to every (completely contractive) representation of the quiver algebra $\mathcal{T}_+(E(\mathcal{Q}))$ we associate a family $\{T(\alpha) : \alpha \in \mathcal{Q}\}$ of maps on H that satisfy $\sum_\alpha T(\alpha)T(\alpha)^* \leq I$ and $T(\alpha) = Q_{r(\alpha)}T(\alpha)Q_{s(\alpha)}$. Conversely, every such family defines a representation by writing $\tilde{T}(f \otimes h) = \sum f(\alpha)T(\alpha)h$. Thus, representations are indexed by such families. Note that, in fact, $(\sigma \times T)(S_\alpha) = T(\alpha)$ and $(\sigma \times T)(P_v) = Q_v$ (where S_α and P_v are as in Example 2.9).*

Remark 2.16 *One of the principal difficulties one faces in dealing with $\mathcal{T}_+(E)$ and $H^\infty(E)$ is to decide when the integrated form, $\sigma \times T$, of a completely contractive covariant representation (T, σ) extends from $\mathcal{T}_+(E)$ to $H^\infty(E)$. This problem arises already in the simplest situation, vis. when $M = \mathbb{C} = E$. In this setting, T is given by a single contraction operator $T(1)$ on a Hilbert space, $\mathcal{T}_+(E)$ "is" the disc algebra and $H^\infty(E)$ "is" the space of bounded analytic functions on the disc. The representation $\sigma \times T$ extends from the disc algebra to $H^\infty(E)$ precisely when there is no singular part to the spectral measure of the minimal unitary dilation of $T(1)$. We are not aware of a comparable result in our general context but we have some sufficient conditions. One of them is given in the following lemma. It is not necessary in general.*

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Lemma 2.17 *([32]) If $\|\tilde{T}\| < 1$ then $\sigma \times T$ extends to a σ-weakly continuous representation of $H^\infty(E)$.*

Other sufficient conditions are presented in Section 7 of [32].

2.2 Duality and point evaluation

The following definition is motivated by condition (2) above.

Definition 2.18 *Let $\sigma : M \to B(H)$ be a normal representation of the von Neumann algebra M on the Hilbert space H. Then for a W^*-correspondence E over M, the σ-dual of E, denoted E^σ, is defined to be*

$$\{\eta \in B(H, E \otimes_\sigma H) \mid \eta\sigma(a) = (\varphi(a) \otimes I)\eta, \ a \in M\}.$$

As we note in the following proposition, the σ-dual carries a natural structure of a W^*-correspondence. The reason to define the σ-dual using covariance condition which is the "adjoint" of condition (2) is to get a *right* W^*-module (instead of a left W^*-module) over $\sigma(M)'$.

Proposition 2.19 *With respect to the actions of $\sigma(M)'$ and the $\sigma(M)'$-valued inner product defined as follows, E^σ becomes a W^*-correspondence over $\sigma(M)'$: For $a, b \in \sigma(M)'$, and $\eta \in E^\sigma$, $a \cdot \eta \cdot b := (I \otimes a)\eta b$, and for $\eta, \zeta \in E^\sigma$, $\langle \eta, \zeta \rangle_{\sigma(M)'} := \eta^*\zeta$.*

Example 2.20 *If $M = E = \mathbb{C}$, H is arbitrary and σ is the representation of $\mathbb{C}$ on H, then $\sigma(M)' = B(H)$ and $E^\sigma = B(H)$.*

Example 2.21 *If Θ is a contractive, normal, completely positive map on a von Neumann algebra M and if $E = M \otimes_\Theta M$ (see Example 2.7) then, for every faithful representation σ of M on H, the σ-dual is the space of all bounded operators mapping H into the Stinespring space K (associated with Θ as a map from M to $B(H)$) that intertwine the representation σ (on H) and the Stinespring representation π (on K). This correspondence was proved very useful in the study of completely positive maps. (See [30], [35] and [31]). If $M = B(H)$ this is a Hilbert space and was studied by Arveson ([8]). Note also that, if Θ is an endomorphism, then this dual correspondence is the space of all operators on H intertwining σ and $\sigma \circ \Theta$.*

We now turn to define point evaluation. Note that, given σ as above, the operators in E^σ whose norm does not exceed 1 are precisely the adjoints of the operators of the form $\tilde{T}$ for a covariant pair (T, σ). In particular, every η in the *open* unit ball of E^σ (written $\mathbb{D}(E^\sigma)$) gives rise to a covariant pair (T, σ) (with $\eta = \tilde{T}^*$) such that $\sigma \times T$ is a representation of $H^\infty(E)$. Given $X \in H^\infty(E)$ we may apply $\sigma \times T$ to it. The resulting operator in $B(H)$ will be denoted by $X(\eta^*)$. That is,

$$X(\eta^*) = (\sigma \times T)(X)$$

where $\tilde{T} = \eta^*$.

In this way, we view every element in the Hardy algebra as a $(B(H)$-valued) function on $\mathbb{D}((E^\sigma)^*)$.

Example 2.22 *Suppose $M = E = \mathbb{C}$ and σ the representation of $\mathbb{C}$ on some Hilbert space H. Then $H^\infty(E) = H^\infty(\mathbb{T})$ and (Example 2.20) E^σ is isomorphic to $B(H)$. If $X \in H^\infty(E) = H^\infty(\mathbb{T})$, so that we may view X with a bounded analytic function on the open disc in the plane, then for $S \in E^\sigma = B(H)$, it is not hard to check that $X(S^*)$, as defined above, is the same as the value provided by the Sz.-Nagy-Foiaş H^∞-functional calculus.*

Note that, for a given $\eta \in \mathbb{D}(E^\sigma)$, the map $X \mapsto X(\eta^*)$ is a σ-weakly continuous homomorphism on the Hardy algebra. Thus, in order to compute $X(\eta^*)$, it suffices to know its values on the generators. This is given in the following (easy to verify) lemma.

Lemma 2.23 *Let σ be a faithful normal representation of M on H and for $\xi \in E$ write L_ξ for the map from H to $E \otimes_\sigma H$ defined by $L_\xi h = \xi \otimes h$. Then, for $\xi \in E$, $a \in M$ and $\eta \in \mathbb{D}(E^\sigma)$,*

(i) *$(T_\xi)(\eta^*) = \eta^* \circ L_\xi$, and*

(ii) *$(\varphi_\infty(a))(\eta^*) = \sigma(a)$*

(Recall that η^* is a map from $E \otimes_\sigma H$ to H.)

A formula for computing $X(\eta^*)$, without referring to the generators, will be presented later (Proposition 2.30).

Example 2.24 *In the setting of Example 2.5 we may identify the Hilbert space $E \otimes_\sigma H = {}_\alpha M \otimes_\sigma H$ with H via the unitary operator mapping $a \otimes h$ (in ${}_\alpha M \otimes_\sigma H$) to $\sigma(a)h$. Using this identification, we may identify E^σ with $\{\eta \in B(H) : \eta\sigma(a) = \sigma(\alpha(a))\eta, \ a \in M\}$.*

Applying the lemma 2.23, we obtain $w(\eta^*) = T_1(\eta^*) = \eta^* \circ L_1 = \eta^*$ (viewed now as an operator in $B(H)$). Thus, if $X = \sum w^n b_n$ (as a formal series), with $b_n = \varphi_\infty(a_n)$ and $\eta \in \mathbb{D}(E^\sigma)$, then

$$X(\eta^*) = \sum (\eta^*)^n \sigma(a_n)$$

with the sum converging in the norm on $B(H)$. (In a sense, this equation asserts that Ceasaro summability implies Abel summability even in this abstract setting.)

Example 2.25 *Let $\mathcal{D}$, U and $E = U\mathcal{D}$ be as in Example 2.6. Let σ be the identity representation of $\mathcal{D}$ on $H = l^2(\mathbb{Z})$. The map $V(UD \otimes_\sigma h) = Dh$ (for $D \in \mathcal{D}, h \in H$) is a unitary operator from $E \otimes_\sigma H$ onto H such that, for every $\eta \in E^\sigma$, $V\eta \in U^*\mathcal{D}$ and, conversely, for every $D \in \mathcal{D}$, $V^*U^*D^*$ lies in E^σ. We write η_D for $V^*U^*D^*$. Recall that the Hardy algebra is $\mathcal{U}$ (the algebra of all upper triangular operators on H). Given $X \in \mathcal{U}$ we shall write X_n for the nth upper diagonal of X. A simple computation shows that, for $D \in \mathcal{D}$ with $\|D\| < 1$,*

$$X(\eta_D^*) = \sum_{n=0}^{\infty} U^n (U^*D)^n X_n.$$

Note here that, in [2], the authors defined point evaluations for operators $X \in \mathcal{U}$. In their setting one evaluates X on the open unit ball of $\mathcal{D}$ and the values are also in $\mathcal{D}$. Their formula (for what in [3] is called the right point evaluation) is

$$X^\Delta(D) = \sum_{n=0}^{\infty} U^n (U^*D)^n X_n U^{*n}.$$

(One can also define a left point evaluation). The apparent similarity of the two formulas above may be deceiving. Note that both their point evaluation and ours can be defined also for "block upper triangular" operators (acting on $l^2(\mathbb{Z}, K)$ for some Hilbert space K). But, in that case, the relation between the two formulas is no longer clear. In fact, our point evaluation is multiplicative (that is, $(XY)(\eta^*) = X(\eta^*)Y(\eta^*)$) while theirs is not. On the other hand,

their point evaluation is "designed" to satisfy the property that, for $X \in \mathcal{U}$ and $D \in \mathcal{D}$, $(X - X^{\Delta}(D))(U - D)^{-1} \in \mathcal{U}$ ([2, Theorem 3.4]) . For our point evaluation (in the general setting), it is not even clear how to state such a property.

Example 2.26 *(Quiver algebras) Let $\mathcal{Q}$ be a quiver as in Example 2.9 and write $E(\mathcal{Q})$ for the associated correspondence. We fix a faithful representation σ of $M = C(V)$ on H. As we note in Example 2.15 , this gives a family $\{Q_v\}$ of projections whose sum is I (and, as σ is faithful, none is 0). Write H_v for the range of Q_v. Then $\sigma(M)' = \oplus_v B(H_v)$ and we write elements there as functions ψ defined on V with $\psi(v) \in B(H_v)$. To describe the σ-dual of E we can use Example 3.4 in [32]. We may also use the description of the maps $\tilde{T}$ in Example 2.15 because every η in the closed unit ball of E^{σ} is $\tilde{T}^*$ for some representation (σ, T) of E. Using this, we may describe an element η of E^{σ} as a family of $B(H)$-valued operators $\{\eta(\beta) : \beta \in \mathcal{Q}^{-1}\}$ where $\mathcal{Q}^{-1}$ is the quiver obtained from $\mathcal{Q}$ by reversing all arrows. The $\sigma(M)'$-module structure of E^{σ} is described as follows. For $\eta \in E^{\sigma}$, $\psi \in \sigma(M)'$ and $\beta \in \mathcal{Q}^{-1}$,*

$$(\eta\psi)(\beta) = \eta(\beta)\psi(s(\beta)),$$

and

$$(\psi\eta)(\beta) = \psi(r(\beta))\eta(\beta).$$

The $\sigma(M)'$-valued inner product is given by the formula

$$\langle \eta, \zeta \rangle (v) = \sum_{s(\beta)=v} \eta(\beta)^* \zeta(\beta),$$

for $\eta, \zeta \in E^{\sigma}$ and $v \in V$.

Recall that the quiver algebra is generated by a set of partial isometries $\{S_\alpha\}$ and projections $\{P_v\}$ (see Example 2.9). If σ is given and $\eta^* = \tilde{T}$ lies in the open unit ball of $(E^{\sigma})^*$ and $\tilde{T}$ is given by a row contraction $(T(\alpha))$ (as in Example 2.15), then the point evaluation for the generators is defined by $S_\alpha(\eta^*) = T(\alpha) = \eta(\alpha^{-1})^*$ and $P_v(\eta^*) = Q_v$. For a general $X \in H^{\infty}(\mathcal{Q})$, $X(\eta^*)$ is defined by the linearity, multiplicativity and σ-weak continuity of the map $X \mapsto X(\eta^*)$.

We turn now to some general results concerning the σ-dual. First, the term "dual" that we use is justified by the following result.

Theorem 2.27 *([32, Theorem 3.6]) Let E be a W^*-correspondence over M and let σ be a faithful, normal representation of M on H. If we write ι for the identity representation of $\sigma(M)'$ (on H) then one may form the ι-dual of E^σ and we have*

$$(E^\sigma)^\iota \cong E.$$

The following lemma summarizes Lemmas 3.7 and 3.8 of [32] and shows that the operation of taking duals behaves nicely with respect to direct sums and tensor products.

Lemma 2.28 *Given W^*-correspondences E, E_1 and E_2 over M and a faithful representation σ of M on H, we have*

(i) *$(E_1 \oplus E_2)^\sigma \cong E_1^\sigma \oplus E_2^\sigma$.*

(ii) *$(E_1 \otimes E_2)^\sigma \cong E_2^\sigma \otimes E_1^\sigma$.*

(iii) *$\mathcal{F}(E)^\sigma \cong \mathcal{F}(E^\sigma)$.*

(iv) *The map $\eta \otimes h \mapsto \eta(h)$ induces a unitary operator from $E^\sigma \otimes_\iota H$ onto $E \otimes_\sigma H$.*

(v) *Applying item (iv) above to $\mathcal{F}(E)$ in place of E, we get a unitary operator U from $\mathcal{F}(E^\sigma) \otimes H$ onto $\mathcal{F}(E) \otimes H$.*

Although $H^\infty(E)$ was defined as a subalgebra of $\mathcal{L}(\mathcal{F}(E))$ it is often useful to consider a (faithful) representation of it on a Hilbert space. Given a faithful, normal, representation σ of M on H we may "induce" it to a representation of the Hardy algebra. To do this, we form the Hilbert space $\mathcal{F}(E) \otimes_\sigma H$ and write

$$Ind(\sigma)(X) = X \otimes I, \ \ X \in H^\infty(E).$$

(in fact, this is well defined for every X in $\mathcal{L}(\mathcal{F}(E))$. Such representations were studied by M. Rieffel in [45]). $Ind(\sigma)$ is a faithful representation and is an homeomorphism with respect to the σ-weak topologies. Similarly one defines $Ind(\iota)$, a representation of $H^\infty(E^\sigma)$. The following theorem shows that, roughly speaking, the algebras $H^\infty(E)$ and $H^\infty(E^\sigma)$ are the commutant of each other.

Theorem 2.29 *[32, Theorem 3.9] With the operator U as in part (v) of Lemma 2.28, we have*

$$U^*(Ind(\iota)(H^\infty(E^\sigma)))U = (Ind(\sigma)(H^\infty(E)))'$$

and, consequently,

$$(Ind(\sigma)(H^\infty(E)))'' = Ind(\sigma)(H^\infty(E)).$$

We may now use the notation set above to present a general formula for point evaluation. For its proof, see [32, Proposition 5.1].

Proposition 2.30 *If σ is a faithful normal representation of M on H, let ι_H denote the imbedding of H into $\mathcal{F}(E^\sigma) \otimes H$ and write P_k for the projection of $\mathcal{F}(E^\sigma) \otimes H$ onto $(E^\sigma)^{\otimes k} \otimes H$. Also, for $\eta \in \mathbb{D}(E^\sigma)$ and $k \geq 1$, note that $\eta^{\otimes k}$ lies in $(E^\sigma)^{\otimes k}$ and that $L^*_{\eta^{\otimes k}}$ maps $(E^\sigma)^{\otimes k} \otimes H$ into H in the obvious way (and, for $k = 0$, this is ι_H). Then, for every $X \in H^\infty(E)$,*

$$X(\eta^*) = \sum_{k=0}^{\infty} L^*_{\eta^{\otimes k}} P_k U^*(X \otimes I)U\iota_H$$

where U is as defined in Lemma 2.28.

3   Nevanlinna-Pick Theorem

Our goal in this section is to present a generalization of the Nevanlinna-Pick theorem. First, recall the classical theorem.

Theorem 3.1 . *Let $z_1, \ldots z_k \in \mathbb{C}$ with $|z_i| < 1$ and $w_1, \ldots w_k \in \mathbb{C}$. Then the following conditions are equivalent.*

(1) *There is a function $f \in H^\infty(\mathbb{T})$ with $\|f\| \leq 1$ such that $f(z_i) = w_i$ for all i.*

(2)

$$\left(\frac{1 - w_i \overline{w_j}}{1 - z_i \overline{z_j}} \right) \geq 0.$$

Since we are able to view elements of $H^\infty(E)$ as functions on the open unit ball of E^σ, it makes sense to seek necessary and sufficient conditions for finding an element $X \in H^\infty(E)$ with norm less or equal 1 whose values at some prescribed points, $\eta_1, \ldots \eta_k$, in that open unit ball are prescribed operators $C_1, \ldots C_k$ in $B(H)$. To state our conditions we need some notation. For operators B_1, B_2 in $B(H)$ we write $Ad(B_1, B_2)$ for the map, from $B(H)$ to itself, mapping S to $B_1 S B_2^*$. Also, for elements η_1, η_2 in $\mathbb{D}(E^\sigma)$, we let θ_{η_1,η_2} denote the map, from $\sigma(M)'$ to itself, that sends a to $\langle \eta_1, a\eta_2 \rangle$. Then our generalization of the Nevanlinna-Pick theorem may be formulated as follows.

Theorem 3.2 *Let σ be a faithful normal representation of M on H. Fix $\eta_1, \ldots \eta_k \in E^\sigma$ with $\|\eta_i\| < 1$ and $B_1, \ldots B_k, C_1, \ldots C_k \in B(H)$. Then the following conditions are equivalent*

(1) *There exists an $X \in H^\infty(E)$ with $\|X\| \le 1$ such that $B_i X(\eta_i^*) = C_i$ for all i.*

(2) *The map from $M_k(\sigma(M)')$ into $M_k(B(H))$ defined by the $k \times k$ matrix*

$$\left((Ad(B_i, B_j) - Ad(C_i, C_j)) \circ (id - \theta_{\eta_i,\eta_j})^{-1} \right)$$

is completely positive.

Remark 3.3 *If $M = B(H)$ (and, then $\sigma(M)' = \mathbb{C}I$), condition (2) becomes*

$$\left(\frac{B_i B_j^* - C_i C_j^*}{1 - \langle \eta_i, \eta_j \rangle} \right) \ge 0.$$

This follows easily from a result of M. D. Choi ([16]).

For the complete proof of Theorem 3.2 we refer the reader to [32, Theorem 5.3]. Here we just remark that in order to prove that (1) implies (2) one uses the complete positivity condition of (2) to construct a subspace $\mathcal{M} \subseteq \mathcal{F}(E^\sigma) \otimes H$ that is invariant under $Ind(\iota)(H^\infty(E^\sigma))^*$ and a contraction R that commutes with the restriction of $Ind(\iota)(H^\infty(E^\sigma))^*$to $\mathcal{M}$. Then it is possible to apply the commutant lifting theorem of [28, Theorem 4.4] to R^* to get a contraction on $\mathcal{F}(E^\sigma) \otimes H$ that commutes with $Ind(\iota)(H^\infty(E^\sigma))$. An application of Theorem 2.29 completes the proof.

The following is a consequence of Theorem 3.2. It may be viewed as a generalization of the classical Schwartz's lemma.

Theorem 3.4 *Suppose an element X of $H^\infty(E)$ has norm at most one and satisfies the equation $X(0) = 0$. Then for every $\eta^* \in \mathbb{D}((E^\sigma)^*)$ the following assertions are valid:*

1. *If a is a nonnegative element in $\sigma(M)'$, and if $\langle \eta, a \cdot \eta \rangle \leq a$, then*

$$X(\eta^*)aX(\eta^*)^* \leq \langle \eta, a \cdot \eta \rangle.$$

2. *If $\eta^{\otimes k}$ denotes the element $\eta \otimes \eta \otimes \cdots \otimes \eta \in E^{\otimes k}$, then*

$$X(\eta^*)\langle \eta^{\otimes k}, \eta^{\otimes k} \rangle X(\eta^*)^* \leq \langle \eta^{\otimes k+1}, \eta^{\otimes k+1} \rangle.$$

3. *$X(\eta^*)X(\eta^*)^* \leq \langle \eta, \eta \rangle$.*

We now illustrate how to apply Theorem 3.2 in various settings.

Example 3.5 *When $M = H = E = \mathbb{C}$, we obtain Theorem 3.1.*

Example 3.6 *If $M = E = \mathbb{C}$ and if H is arbitrary, then $E^\sigma = B(H)$ and Theorem 3.2 yields the following result.*

Theorem 3.7 *Given $T_1, \ldots T_k \in B(H), \|T_i\| < 1$ and $B_1, \ldots, B_k, C_1, \ldots C_k$ in $B(H)$. then the following conditions are equivalent.*

(1) *There exists a function $f \in H^\infty(\mathbb{T})$ with $\|f\| \leq 1$ and $B_i f(T_i) = C_i$.*

(2) *The map defined by the matrix (ϕ_{ij}) is completely positive where*

$$\phi_{ij}(A) = \sum_{k=0}^{\infty}(B_i T_i^k A T_j^{*k} B_j^* - C_i T_i^k A T_j^{*k} B_j).$$

Example 3.8 *Assume $M = B(H) = E$. Then $M' = \mathbb{C}I$ and $E^\sigma = \mathbb{C}$ and Theorem 3.2 specializes to the following.*

Theorem 3.9 *Given $z_1, \ldots z_k \in \mathbb{D}$ and $B_1, \ldots, B_k, C_1, \ldots C_k$ in $B(H)$, then the following conditions are equivalent.*

(1) *There exists $G \in H^\infty(\mathbb{T}) \otimes B(H)$ with $\|G\| \leq 1$ such that $B_i G(z_i) = C_i$ for all i.*

(2)

$$\left(\frac{B_i B_j^* - C_i C_j^*}{1 - z_i \overline{z_j}}\right) \geq 0.$$

Example 3.10 *Set $M = B(H)$ and $E = C_n(B(H))$ (that is, E is a column of n copies of $B(H)$). Then $M' = \mathbb{C}I$, $E^\sigma = \mathbb{C}^n$ and Theorem 3.2 yields the following theorem due to Davidson and Pitts [17], Arias and Popescu [6] and Popescu [44].*

Theorem 3.11 *Given $\eta_1, \ldots \eta_k$ in the open unit ball of $\mathbb{C}^n$ and $C_1, \ldots C_k \in B(H)$, then the following conditions are equivalent.*

(1) *There is a $Y \in B(H) \otimes \mathcal{L}_n$ with $\|Y\| \leq 1$ such that $(\eta_i^* \times id)(Y) = C_i$ for all i.*

(2)

$$\left(\frac{I - C_i C_j^*}{1 - \langle \eta_i, \eta_j \rangle}\right) \geq 0.$$

Moreover, if, for all i, the C_i all lie in some von Neumann subalgebra $N \subseteq B(H)$, then Y can be chosen in $N \otimes \mathcal{L}_n$.

Our final example of this section concerns interpolation for nest algebras. The first interpolation result for nest algebras was proved by Lance ([23]). It was later generalized by Anoussis ([5]) and by Katsoulis, Moore and Trent ([21]). A related result was proved by Ball and Gohberg ([11]). The results we present below recapture the results of [21].

Theorem 3.12 *Let $\mathcal{N}$ be a nest of projections in $B(H)$ and fix B, C in $B(H)$. Then the following conditions are equivalent.*

(1) *There exists an $X \in Alg\mathcal{N}$ with $\|X\| \leq 1$ and $BX = C$.*

(2) *For all projections $N \in \mathcal{N}$, $CNC^* \leq BNB^*$.*

The "vector version" of this theorem is the following.

Corollary 3.13 *Let $\mathcal{N}$ be a nest in $B(H)$ and fix $u_1, \ldots u_k, v_1, \ldots v_k$ in H. Then the following conditions are equivalent.*

(1) *There exists $X \in Alg\mathcal{N}$ with $\|X\| \leq 1$ and $Xu_i = v_i$ for all i.*

(2) For all $N \in \mathcal{N}$,

$$\left(\langle N^{\perp} v_i, N^{\perp} v_j \rangle \right) \leq \left(\langle N^{\perp} u_i, N^{\perp} u_j \rangle \right),$$

where $N^{\perp}$ denotes $I - N$.

These results are not immediate corollaries of Theorem 3.2 because, for a general nest $\mathcal{N}$, $Alg\mathcal{N}$ is not of the form $H^{\infty}(E)$. However, when $\mathcal{N}$ is finite, $Alg\mathcal{N}$ is a Hardy Algebra by Example 2.8. In this case, the conclusions are fairly straight forward computations. The case of general nests is then handled by approximation techniques along the lines of [23] and [7]. Full details may be found in [32, Theorem 6.8 and Corollary 6.9].

4   Schur class operator functions and realization

In this section we relate the complete positivity condition of Theorem 3.2 to the concept of a Schur class function. As mentioned in the introduction, this may be viewed as part of a general program to find equivalences between canonical model theory, "non commutative" systems theory and scattering theory. The results below are proved in [34].

We start with the following definition.

Definition 4.1 *Let S be a set, A and B be two C^*-algebras and write $\mathcal{B}(A, B)$ for the space of bounded linear maps from A to B. A function*

$$K : S \times S \to \mathcal{B}(A, B)$$

will be called a completely positive definite kernel *(or a* CPD-kernel *) if, for all choices of $s_1, \ldots, s_k$ in S, the map*

$$K^{(k)} : (a_{ij}) \mapsto (K^{(k)}(s_i, s_j)(a_{ij}))$$

from $M_k(A)$ to $M_k(B)$ is completely positive.

This concept of CPD-kernels was studied in [14] (see, in particular, Lemma 3.2.1 there for conditions on K that are equivalent to being a CPD-kernel).

Definition 4.2 *Let $\mathcal{E}$ be a Hilbert space and $Z : \mathbb{D}((E^\sigma)^*) \to B(\mathcal{E})$ be a $B(\mathcal{E})$-valued function. Then Z is said to be a* Schur class operator function *if*

$$K(\eta^*, \zeta^*) = (id - Ad(Z(\eta^*), Z(\zeta^*)) \circ (id - \theta_{\eta,\zeta})^{-1}$$

is a CPD-kernel on $\mathbb{D}((E^\sigma)^*)$. (We use here the notation set for Theorem 3.2).

Note that, when $M = E = B(\mathcal{E})$ and σ is the identity representation of $B(\mathcal{E})$ on $\mathcal{E}$, $\sigma(M)'$ is $\mathbb{C}I_\mathcal{E}$, E^σ is isomorphic to $\mathbb{C}$ and $\mathbb{D}((E^\sigma)^*)$ may be identified with the open unit ball $\mathbb{D}$ of $\mathbb{C}$. In this case the definition above recovers the classical Schur class functions. More precisely, these functions are usually defined as analytic functions Z from $\mathbb{D}$ into the closed unit ball of $B(\mathcal{E})$ but it is known that this is equivalent to the positivity of the Pick kernel $k_Z(z, w) = (I - Z(z)Z(w)^*)(1 - z\bar{w})^{-1}$. The argument of [32, Remark 5.4] shows that the positivity of this kernel is equivalent, in this case, to the condition of Definition 4.2.

Note that it follows from Theorem 3.2 that every operator in the closed unit ball of $H^\infty(E)$ determines (by point evaluation) a Schur class operator function. In fact we have the following result whose proof may be found in [34].

Theorem 4.3 *([34]) Let M be a von Neumann algebra, let E be a W^*-correspondence over M and let σ be a faithful normal representation of M on a Hilbert space $\mathcal{E}$. For a function $Z : \mathbb{D}((E^\sigma)^*) \to B(\mathcal{E})$, the following conditions are equivalent.*

(1) *Z is a Schur class operator function.*

(2) *There is an X in the closed unit ball of $H^\infty(E)$ such that $X(\eta^*) = Z(\eta^*)$ for all $\eta \in \mathbb{D}(E^\sigma)$.*

(3) *(Realization) There is a Hilbert space H, a normal representation τ of $N := \sigma(M)'$ on H and operators A, B, C and D such that*

    (i) *$A \in B(\mathcal{E}), B \in B(H, \mathcal{E})$, $C \in B(\mathcal{E}, H)$ and $D \in B(H, E^\sigma \otimes H)$.*

    (ii) *A, B, C and D intertwine the actions of N (on the relevant spaces).*

(iii) The operator

$$V := \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{pmatrix} \mathcal{E} \\ H \end{pmatrix} \rightarrow \begin{pmatrix} \mathcal{E} \\ E^\sigma \otimes H \end{pmatrix}$$

is a coisometry.

(iv) For every $\eta \in \mathbb{D}(E^\sigma)$,

$$Z(\eta^*) = A + B(I - L_\eta^* D)^{-1} L_\eta^* C$$

where $L_\eta : H \rightarrow E^\sigma \otimes H$ is defined by $L_\eta h = \eta \otimes h$.

Note that X in part (2) of the Theorem is not necessarily unique. (Although, as shown in [34], it is possible to choose σ such that the choice of X will be unique).

One may apply the techniques developed (in [34]) for the proof of the Theorem 4.3 to establish the following extension result.

Proposition 4.4 *Every function defined on a subset Ω of the open unit ball $\mathbb{D}((E^\sigma)^*)$ with values in some $B(\mathcal{E})$ such that the associated kernel (defined on $\Omega \times \Omega$) is a CPD-kernel may be extended to a Schur class operator function (defined on all of $\mathbb{D}((E^\sigma)^*))$.*

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